**Cross Product**

The cross product (also known as the vector product) is an operation utilizing two vectors in the three-dimensional plane. When given two vectors $\vec{n}$ and $\vec{m}$, the cross product of these vectors is the vector that is perpendicular (or orthogonal) to both $\vec{n}$ and $\vec{m}$. Like the dot product, it is the multiplication of vectors, however the cross product takes into account the orientation of the vectors. The direction of the resulting vector is given by the right-hand rule (shown in the image below). While similar to the dot product, the difference between the two is that **the dot product produces a number while the cross product produces a vector**. Cross product has applications in calculating angular momentum and torque in terms of space travel and exploration, but can also be used to determine the area of parallelograms formed by vectors.



Like dot product, there are two equations that can be used to calculate the cross product. The first formula can be difficult to remember, but can be easy using memory tricks or by understanding the derivation (which requires the use of matrices).

When given two vectors $\vec{n}$ and $\vec{m}$, the cross product of these vectors is determined by:
$\vec{n}$x$\vec{m}$= < $\vec{n}$2$\vec{m}$3 - $\vec{n}$3$\vec{m}$2, $\vec{n}$3$\vec{m}$1- $\vec{n}$1$\vec{m}$3, $\vec{n}$1$\vec{m}$2- $\vec{n}$2$\vec{m}$1>
or
$\vec{n}$x$\vec{m}$= $\vec{i}$($\vec{n}$2$\vec{m}$3 - $\vec{n}$3$\vec{m}$2) + $\vec{j}$($\vec{n}$3$\vec{m}$1- $\vec{n}$1$\vec{m}$3) + $\vec{k}$($\vec{n}$1$\vec{m}$2- $\vec{n}$2$\vec{m}$1)

Order is important, which makes the formula is difficult to remember at first, so here are some tips for memorizing:
- Consider the vector components as numbers. Since the vectors maintain their order (always the $\vec{n}$ component multiplied by the $\vec{m}$ component for $\vec{n}$x$\vec{m}$), the only concern is which components are being multiplied.
- Only 3 pairs of numbers need to be remembered, since the subtraction is simply the numbers in the pairs reversed.
- One trick to remembering the pairs is to start with the numbers 123. Starting from 2 and moving left once we have 2 and 3. Next we have 3 and 1. Finally the last pair is 1 and 2.

For a more intuitive understanding of the formula, we look at its derivation. There are two ways to derive the formula and both require the fact that the cross product is the determinant of a 3x3 matrix. Knowledge of matrices isn’t required at the moment, but a simple understanding is beneficial.



Paul’s Online Notes has a wonderful summary of the derivation, please look over that derivation here: <http://tutorial.math.lamar.edu/Classes/CalcII/CrossProduct.aspx>

For additional assistance in understanding the derivation of the formula using matrices, a video link has been provided on the activity page.

Example
Given vectors $\vec{n}$ =<-1, 3, -2> and $\vec{m}$= <4, 1, 2> ,
Calculate:
a) $\vec{n}$x$\vec{m}$ b) $\vec{m}$x$\vec{n}$
c) $\vec{n}$x$\vec{n}$ d) $\vec{m}$x$\vec{m}$

a) $\vec{n}$x$\vec{m}$= <(3)(2) – (1)(-2), (-2)(4) – (-1)(2), (-1)(1) – (3)(4)>
 $\vec{n}$x$\vec{m}$= <6 – (-2), (-8) – (-2), (-1) – 12>
 $\vec{n}$x$\vec{m}$= <8, -6, -13>

b) ) $\vec{m}$x$\vec{n}$= <(1)(-2) – (3)(2), (-1)(2) – (-2)(4), (3)(4) – (-1)(1)>
 $\vec{m}$x$\vec{n}$= <(-2) – 6, (-2) – (-8), 12 – (-1)>
 $\vec{m}$x$\vec{n}$= <-8, 6, 13>

c) $\vec{n}$x$\vec{n}$= <(3)(2) – (2)(3), (-2)(4) – (-2)(4), (-1)(1) – (-1)(1)>
 $\vec{n}$x$\vec{n}$= <6 – 6, (-8) – (-8), (-1) – (-1)>
 $\vec{n}$x$\vec{n}$= <0, 0, 0>
d) From our solution in question c, we can know that $\vec{m}$x$\vec{m}$=<0, 0, 0>

From the examples, we can see a few properties of the cross product namely that $\vec{n}$x$\vec{m}$= -($\vec{m}$x$\vec{n}$) and that $\vec{n}$x$\vec{n}$= $\vec{0}$

The second formula for the cross product can be derived from the fact that the magnitude of the cross product can be used to find the area of a parallelogram.



Consider the parallelogram above. The magnitudes of the vectors represent the lengths of the sides, and since the cross product of the vectors equals the area, we have that A= I$\vec{a}$x$\vec{b}$I

We know that rectangles and parallelograms have the same area, and so the area of the parallelogram can also be expressed as A= h I$\vec{a}$I. Therefore, it can be said that I$\vec{a}$x$\vec{b}$I= h I$\vec{a}$I.

Using trigonometric relationships, we know that h/(I$\vec{b}$I)= sinθ. Rearranging the equation, we find that h=I$\vec{b}$I sinθ. Combining this equation along with the one determined before, we have the cross product magnitude formula.

I$\vec{a}$x$\vec{b}$I= I$\vec{a}$I I$\vec{b}$I sinθ

Properties of Cross product

Given vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and scalar n we have:

1. $\vec{a}$x$\vec{b}$= -$\vec{b}$x$\vec{a}$
2. $\vec{a}$x$\vec{a}$ = $\vec{0}$
3. For collinear/parallel vectors, θ= 0 and sin(0)= 0. Therefore, I$\vec{a}$x$\vec{b}$I= I$\vec{a}$I I$\vec{b}$I sin(0)= 0
4. $\vec{c}$x($\vec{a}$+$\vec{b}$) = $\vec{c}$x$\vec{a}$+$\vec{c}$x$\vec{b}$
5. (n$\vec{a}$)x$\vec{b}$= $\vec{a}$x(n$\vec{b}$)= n($\vec{a}$x$\vec{b}$)
6. $\vec{c}$•($\vec{a}$x$\vec{b}$) =($\vec{c}$x$\vec{b}$)•$\vec{a}$

Knowing that the area of a four-sided figure is I$\vec{a}$x$\vec{b}$I, we can derive the volume of a three-dimensional figure or parallelepiped.



We begin with the parallelepiped comprised of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{h}$. We know V= I$\vec{h}$II$\vec{a}$x$\vec{b}$I for a standard rectangular prism. Now have to take into account the fact that vector $\vec{h}$ is not a vector that comprises the parallelepiped. Using trigonometric relationships, we have that I$\vec{h}$I/I$\vec{c}$I= cosθ. Re-arranging the formula we have I$\vec{h}$I=I$\vec{c}$I cosθ.

Substituting I$\vec{h}$I=I$\vec{c}$I cosθ into V= I$\vec{h}$II$\vec{a}$x$\vec{b}$I, we have:

V= I$\vec{a}$x$\vec{b}$I I$\vec{c}$I cosθ
This is similar to the dot product of ($\vec{a}$x$\vec{b}$) and $\vec{c}$. Therefore,
V= I($\vec{a}$x$\vec{b}$)•$\vec{c}$I

As volume cannot be a negative value, we take the absolute value of the result.

From the equation of the volume, we can determine if three vectors lie in the same plane. If three vectors lie in the same plane, the volume would equal zero.

Example

Determine if the following three vectors lie in the same plane. $\vec{a}$= <2, -1, 4>, $\vec{b}$= <1, 4, -7> and $\vec{c}$= <0, -9, 18>

As mentioned before, to determine if the three vectors lie in the same plane, we determine the volume. However, we do not need the absolute value as if the vectors do lie in the same plane, V=0.
V= ($\vec{a}$x$\vec{b}$)•$\vec{c}$

$\vec{a}$x$\vec{b}$= $\vec{i}$ [(-1)(-7)-(4)(4)]+$ \vec{j}$[(4)(1)-(2)(-7)]+$\vec{k}$[(2)(4)-(-1)(1)]
$\vec{a}$x$\vec{b}$= $\vec{i}$ (7-16)+$ \vec{j}$(4-(-14))+$\vec{k}$(8-(-1))
$\vec{a}$x$\vec{b}$= -9$\vec{i}$ +18$\vec{j}$ +9$\vec{k}$ or <-9, 18, 9>

V= ($\vec{a}$x$\vec{b}$)•$\vec{c}$
V= <-9, 18, 9>•<0, -9, 18>
V= (-9)(0)+ (18)(-9)+ (9)(18)
V= 0- 162+ 162
V= 0

Since V= 0, the three vectors lie on the same plane.

Practice Problems

1. Determine the cross product of the vectors:
a) <2, -3, 5> and <0, -1, 4> = <-7, -8, -2>
b) <2, -1, 3> and <3, -1, 2> = <1, 5, 1>
c) <5, -1, 1> and <2, 4, 7> = <-11, -33, 22>
2. Determine the area of the parallelogram formed by the vectors $\vec{a}$= <-1, 3, -2> and $\vec{b}$= <4, 1, 2>
A= I$\vec{a}$x$\vec{b}$I
$\vec{a}$x$\vec{b}$= <8, -6, -13>
A= √82+ (-6)2+ (-13)2
A= √269= 16.4
3. If <-1, 3, 5> x <0, a, 1>= <-2, 1, -1>, what is the value of a?
a= 1
4. If O(0, 0, 0), A(1, 1, 1), B(2, -1, 3) and C(-2, 3, 1) are 4 corners of a parallelepiped so that OA, OB and OC are three edges. Calculate the volume of this parallelepiped.
V= I($\vec{OA}$x$\vec{OB}$)•$\vec{OC}$I
$\vec{OA}$x$\vec{OB}$= <4, -1, -3>
($\vec{OA}$x$\vec{OB}$)•$\vec{OC}$= <4, -1, -3>•<-2, 3, 1>
($\vec{OA}$x$\vec{OB}$)•$\vec{OC}$= 4(-2)+ (-1)(3)+ (-3)(1)= 14
V= I14I= 14
5. Do the vectors $\vec{a}$= $\vec{i}$ +2$\vec{j}$ +$\vec{k}$, $\vec{b}$= 2$\vec{i}$ +4$\vec{j}$ +2$\vec{k}$ and $\vec{c}$= 3$\vec{i}$ -$\vec{j}$ +5$\vec{k}$ lie in the same plane?
$\vec{a}$=< 1, 2, 1>, $\vec{b}$=<2, 4, 2>, $\vec{c}$=<3, -1, 5>
V= ($\vec{a}$x$\vec{b}$)•$\vec{c}$
$\vec{a}$x$\vec{b}$= <(2)(2)-(1)(4), (1)(2)- (1)(2), (1)(4)- (2)(2)>
$\vec{a}$x$\vec{b}$= <0, 0, 0>
($\vec{a}$x$\vec{b}$)•$\vec{c}$= <0, 0, 0>•<3, -1, 5>
($\vec{a}$x$\vec{b}$)•$\vec{c}$= (0)(3)+ (0)(-1)+ (0)(5) = 0
V= 0

Since V=0, the three vectors lie in the same plane.